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Quantum operations fixing a convex cone of density operators on $\mathcal{T}(H)$

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Abstract

The unital quantum operation acting on infinite dimensional quantum states fixing a convex cone of density operators is completely characterized. Based on this result, we classify the commutativity of two quantum operations and determine what kind of measurement statistics are preserved by a unital quantum operation. Our work extends the results of Lee *et al* (2013 *J. Phys. A: Math. Theor.* **46** 205305) to the infinite dimensional case and also corrects part (a) of theorem 6 in Lee *et al* (2013 *J. Phys. A: Math. Theor.* **46** 205305).

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1. Introduction

Quantum processes, also known as quantum operations, quantum channels, or trace-preserving completely positive maps on the trace-class operators, are central to the theory and practice of quantum information processing [2, 3]. They describe how quantum states evolve over a period of time in the presence of noise, or how a device's output depends on its input. They are also complex and unwieldy; to fully specify a quantum process on an n dimensional system requires n^4 real numbers. Thus characterizing quantum operations and studying their properties are two important areas of research in quantum information science [4–19]. In [1], originated from the measurement statistics preservation and fixed point problem of unital quantum operations, the authors gave a necessary and sufficient condition for a convex cone of positive semidefinite matrices to be fixed by a unital quantum operation acting on finite dimensional quantum states. Some applications in quantum information processing are also given. The purpose of this paper

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is to classify quantum operations acting on infinite dimensional quantum states fixing a convex cone of density operators. As applications, we completely characterize the commutativity of two quantum operations and determine what kind of measurement statistics are preserved by a unital quantum operation.

The notations used in this paper are as follows. Let H be a separable infinite dimensional complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$. $\mathcal{B}(H)$ and $\mathcal{T}(H)$ denote the von Neumann algebra of all bounded linear operators and the trace-class operators with the trace norm $\|T\|_1 = \mathrm{Tr}((T^\dagger T)^{\frac{1}{2}}) < +\infty$, respectively. The trace-class positive operators on H with unit trace are called density operators or quantum states. Mathematically, a quantum operation is a trace-preserving completely positive linear map on $\mathcal{T}(H)$. By the classical Kraus representation theorem, a quantum operation can be represented in an elegant form known as the operator-sum representation [2]. That is, Φ is an operation if and only if there exist bounded linear operators $A_k, k \in K$ (a finite or countably infinite index set), on H satisfying $\sum_k A_k^\dagger A_k = I$ such that

$$\Phi(T) = \sum_k A_k T A_k^\dagger$$

holds for every trace-class operator $T \in \mathcal{T}(H)$. If the index set K is infinite, the above infinite sum converges in trace norm. If $\sum_k A_k A_k^\dagger = I$, then Φ is called unital. With the fixed convex cone of a positive semidefinite operator problem [1], we refer to the problem of determining this if Φ fixes a convex cone $\mathcal{C}^+$ formed by the set of all density operators whose supports are contained in $\mathcal{S}$ (a proper closed subspace of H) in the sense that $\Phi(\mathcal{C}^+) \subseteq \mathcal{C}^+$. Here, for a density operator $\rho \in \mathcal{T}(H)$, the support of ρ means the norm closure of the range of ρ. In this case, we also say that Φ fixes $\mathcal{S}$ or $\mathcal{S}$ is an invariant subspace of Φ. If both $\mathcal{S}$ and $\mathcal{S}^\perp$ are invariant subspaces of Φ, we say $\mathcal{S}$ is a reducing subspace of Φ. In this paper, we are only interested in the action of Φ on density operators because if an operator $\sigma \in \mathcal{T}(\mathcal{H})$ describes a physical quantum state, then σ is a density operator.

2. Structural characterization of unital quantum operations acting on infinite dimensional density operators

This section is devoted to giving a necessary and sufficient condition for a convex cone of density operators to be fixed by a unital quantum operation Φ acting on infinite dimensional quantum states. Recall that if $A \in \mathcal{B}(H)$ and $\mathcal{M}$ is a closed subspace of H, we say $\mathcal{M}$ is an invariant subspace for A if $A(x) \in \mathcal{M}$ whenever $x \in \mathcal{M}$. In other words, $A(\mathcal{M}) \subseteq \mathcal{M}$. We say $\mathcal{M}$ is a reducing subspace for A if $A(\mathcal{M}) \subseteq \mathcal{M}$ and $A(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp$, where $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$. It is easy to see that $\mathcal{M}$ is A_k-invariant if and only if $\mathcal{M}^\perp$ is $A_k^\dagger$-invariant. We start with the following lemma which is borrowed from [6, theorem 1.1]. We remark that although it is proved in the finite dimensional case, the proof of this theorem in [6] holds true for the infinite dimensional case. For completeness and for the convenience of readers, we provide a proof.

Lemma 2.1. *Let H be a separable infinite dimensional Hilbert space. Let $\Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a completely positive unital map such that $\Psi(T) = \sum_k A_k T A_k^\dagger$ for every $T \in \mathcal{B}(H)$. If P is a projection operator in $\mathcal{B}(H)$, then we have the following equivalences for the range space $Ran(P)$ of the projection:*

(i) $\Psi(P) \geqslant P$ if and only if $Ran(P)$ is $A_k^\dagger$-invariant for all k,
(ii) $\Psi(P) \leqslant P$ if and only if $Ran(P)$ is A_k-invariant for all k,
(iii) $\Psi(P) = P$ if and only if $Ran(P)$ is A_k-reducing for all k.

Proof. Firstly, we note that the equivalences (i) and (ii) are duals of each other. Indeed, since Ψ is unital, $\Psi(P) \geqslant P \Leftrightarrow \Psi(I-P) \leqslant I-P$, and $Ran(P)$ is $A_k^\dagger$-invariant if and only if $Ran(P)^\perp = Ran(I-P)$ is A_k-invariant. Secondly, (iii) is an immediate result of (i) and (ii). Thus we only prove (ii).

($\Rightarrow$): For each unit vector $x \in Ran(P)$, then $|x\rangle\langle x|$ is a pure state. Since Ψ is order preserving, $|x\rangle\langle x| \leqslant P$ implies $\Psi(|x\rangle\langle x|) = \sum A_k|x\rangle\langle x|A_k^\dagger \leqslant \Psi(P) \leqslant P$. It follows that $A_k|x\rangle\langle x|A_k^\dagger \leqslant P$. Thus $A_k(x) \in Ran(P)$, i.e., $Ran(P)$ is A_k-invariant.

($\Leftarrow$): If $Ran(P)$ is A_k-invariant, then A_k can be written in matrix form as

$$\begin{pmatrix} B_k & C_k \\ 0 & D_k \end{pmatrix}$$

with respect to space decomposition $H = Ran(P) \oplus Ran(P)^\perp$. Therefore the relation $\Psi(I) = \sum_k A_k A_k^\dagger = I$ yields the identity

$$\sum_k \left(B_k B_k^\dagger + C_k C_k^\dagger\right) = P.$$

Thus upon writing $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ with respect to this space decomposition, we can obtain

$$\Psi(P) = \begin{pmatrix} \sum_k B_k B_k^\dagger & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} \sum_k \left(B_k B_k^\dagger + C_k C_k^\dagger\right) & 0 \\ 0 & 0 \end{pmatrix} = P.$$

□

Our main result in this section reads as follows:

Theorem 2.2. *Let Φ be a unital quantum operation on $\mathcal{T}(H)$ and $\mathcal{S}$ be a proper closed subspace of H. If $P_{\mathcal{S}}$ denotes the projection operator onto $\mathcal{S}$, then we have the following equivalences according to the dimensions of $\mathcal{S}$ and $\mathcal{S}^\perp$.*

(i) If $\dim \mathcal{S} < +\infty, \dim \mathcal{S}^\perp = +\infty$, then Φ fixes $\mathcal{S} \Leftrightarrow \Phi(P_{\mathcal{S}}) = P_{\mathcal{S}} \Leftrightarrow Ran(P_{\mathcal{S}})$ is A_k-reducing for all k.

(ii) If $\dim \mathcal{S} = +\infty, \dim \mathcal{S}^\perp < +\infty$, then Φ fixes $\mathcal{S} \Leftrightarrow \Phi(I-P_{\mathcal{S}}) = I-P_{\mathcal{S}} \Leftrightarrow Ran(P_{\mathcal{S}})$ is A_k-reducing for all k.

(iii) If $\dim \mathcal{S} = +\infty, \dim \mathcal{S}^\perp = +\infty$, then Φ fixes $\mathcal{S} \Leftrightarrow \hat{\Phi}(P_{\mathcal{S}}) \leqslant P_{\mathcal{S}}, \hat{\Phi}(I-P_{\mathcal{S}}) \geqslant I-P_{\mathcal{S}} \Leftrightarrow Ran(P_{\mathcal{S}})$ is A_k-invariant for all k, where $\hat{\Phi}(\cdot)$ is the only continuous linear extension of Φ to $\mathcal{B}(H)$ in the strong operator topology.

Proof of theorem 2.2. By lemma 2.1, we need only to prove the first equivalence in each of (i), (ii), (iii) and the proof is divided into four steps.

Step 1. Φ extends continuously in strong operator topology to a unital, trace-preserving, completely positive map $\hat{\Phi}$ on $\mathcal{B}(H)$.

Define naturally $\hat{\Phi} : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\hat{\Phi}(T) = \sum_k A_k T A_k^\dagger$$

for $T \in \mathcal{B}(H)$. For each $0 \leqslant T \leqslant I$, from $0 \leqslant A_k T A_k^\dagger \leqslant A_k A_k^\dagger$ and $\sum_k A_k A_k^\dagger = I$, it follows that $0 \leqslant \hat{\Phi}(T) \leqslant I$. This implies $\hat{\Phi}$ is well-defined on $\mathcal{B}(H)$. Note that $\mathcal{T}(H)$ is a dense subset of $\mathcal{B}(H)$ in the strong operator topology; we need only to show $\hat{\Phi}$ is continuous in the strong operator topology. Let l_2 denote the Hilbert space of column vectors of square summable complex sequences and let $l_2(H) = l_2 \otimes H$ be the Hilbert space of column vectors of square summable sequences with elements in H. Let $V : H \to l_2(H)$ be the linear operator defined by

$$V|h\rangle = \begin{pmatrix} A_1^\dagger|h\rangle \\ A_2^\dagger|h\rangle \\ \vdots \end{pmatrix}.$$

Now V is isometric because

$$\|V|h\rangle\|^2 = \sum_k \langle h|\big(A_k A_k^\dagger\big)|h\rangle = \langle h|\bigg(\sum_k A_k A_k^\dagger\bigg)|h\rangle = \||h\rangle\|^2.$$

Then the adjoint $V^\dagger \in \mathcal{B}(l_2(H), H)$ is given by

$$V^\dagger \begin{pmatrix} |h_1\rangle \\ |h_2\rangle \\ \vdots \end{pmatrix} = \sum_k A_k|h_k\rangle.$$

It follows that $\hat{\Phi}(T) = V^\dagger(I \otimes T)V$ for $T \in \mathcal{B}(H)$, here $I \otimes T = \begin{pmatrix} T & 0 & 0 & \cdots \\ 0 & T & 0 & \cdots \\ 0 & 0 & T & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.
Because the map $T \mapsto (I \otimes T)$ from $\mathcal{B}(H)$ to $\mathcal{B}(l_2 \otimes H)$ is strongly continuous, see p146 in [20], $\hat{\Phi}$ is strongly continuous. From $\sum_k A_k A_k^\dagger = I$, it is clear that $\hat{\Phi}(I) = I$.
Step 2. If $\dim \mathcal{S} < +\infty$, then Φ fixes $\mathcal{S}$ if and only if $\Phi(P_{\mathcal{S}}) = P_{\mathcal{S}}$.
We start with an observation: for each positive operator T on H, the support of T is in $\mathcal{S}$ if and only if $TP_{\mathcal{S}} = P_{\mathcal{S}}T = T$. This observation can be seen from

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

according the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^\perp$.
$(\Rightarrow)$: Φ fixes $\mathcal{S}$ implies the support of $\Phi(P_{\mathcal{S}}/\dim \mathcal{S})$ is in $\mathcal{S}$. From our observation,

$$\Phi(P_{\mathcal{S}})P_{\mathcal{S}} = P_{\mathcal{S}}\Phi(P_{\mathcal{S}}) = \Phi(P_{\mathcal{S}}).$$

By $I = \hat{\Phi}(I) = \hat{\Phi}(P_{\mathcal{S}}) + \hat{\Phi}(I - P_{\mathcal{S}})$, we have

$$1 = \langle x|\hat{\Phi}(P_{\mathcal{S}})|x\rangle + \langle x|\hat{\Phi}(I - P_{\mathcal{S}})|x\rangle$$

for every unit vector $x \in \mathcal{S}$. Note that $\hat{\Phi}(I - P_{\mathcal{S}}) \geqslant 0$, thus

$$0 \leqslant \langle x|\hat{\Phi}(P_{\mathcal{S}})|x\rangle = \langle x|\Phi(P_{\mathcal{S}})|x\rangle \leqslant 1 = \langle x|P_{\mathcal{S}}|x\rangle.$$

Since Φ fixes $\mathcal{S}$, then $\Phi(P_{\mathcal{S}}) \leqslant P_{\mathcal{S}}$, so $P_{\mathcal{S}} - \Phi(P_{\mathcal{S}}) \geqslant 0$. Since $P_{\mathcal{S}}$ is by assumption finite dimensional, and Φ is trace preserving, then $\mathrm{tr}(P_{\mathcal{S}} - \Phi(P_{\mathcal{S}})) = 0$, so $P_{\mathcal{S}} - \Phi(P_{\mathcal{S}}) = 0$.
$(\Leftarrow)$: Let ρ be a density operator whose support is in $\mathcal{S}$. Since Φ is order preserving, $\rho \leqslant P_{\mathcal{S}}$ implies that $\Phi(\rho) \leqslant \Phi(P_{\mathcal{S}}) = P_{\mathcal{S}}$. Hence the support of $\Phi(\rho)$ is in $\mathcal{S}$.
Step 3. If $\dim \mathcal{S}^\perp < +\infty$, then Φ fixes $\mathcal{S} \Leftrightarrow \Phi(I - P_{\mathcal{S}}) = I - P_{\mathcal{S}}$.
$(\Rightarrow)$: Since $\dim \mathcal{S} = +\infty$, we can express $P_{\mathcal{S}} = \sum_{i=1}^{+\infty} |x_i\rangle\langle x_i|$, here $\{|x_i\rangle\}$ is an orthonormal basis of $\mathcal{S}$. Since the support of $|x_i\rangle\langle x_i|$ is in $\mathcal{S}$, the support of $\Phi(|x_i\rangle\langle x_i|)$ is in $\mathcal{S}$. Let $P_k = \sum_{i=1}^{k} |x_i\rangle\langle x_i|$, then $\{P_k\}_k$ converges to $P_{\mathcal{S}}$ in strong operator topology. Noting that $\hat{\Phi}$ is continuous in strong operator topology, we know the support of $\hat{\Phi}(P_{\mathcal{S}})$ is also in $\mathcal{S}$. Therefore $\hat{\Phi}(P_{\mathcal{S}}) = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix}$ according to the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^\perp$.
For arbitrary vectors $|x\rangle \in \mathcal{S}^\perp$, $|y\rangle \in \mathcal{S}$,

$$\langle x|\hat{\Phi}(I - P_{\mathcal{S}})|y\rangle = \langle x|(I - \hat{\Phi}(P_{\mathcal{S}})|y\rangle = \langle x|y - \hat{\Phi}(P_{\mathcal{S}})|y\rangle = 0.$$

This implies that $\hat{\Phi}(I-P_{\mathcal{S}}) = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}$ according to the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^{\perp}$. From $\hat{\Phi}(I-P_{\mathcal{S}}) + \hat{\Phi}(P_{\mathcal{S}}) = I$, we obtain $T_{22} = I - P_{\mathcal{S}}$. Since Φ is trace preserving, we have $\mathrm{tr}(\hat{\Phi}(I-P_{\mathcal{S}})) = \mathrm{tr}(\Phi(I-P_{\mathcal{S}})) = \mathrm{tr}(I-P_{\mathcal{S}})$. Thus $\mathrm{tr}(T_{11}) = 0$ and so $T_{11} = 0$. Therefore, we have $\mathrm{Ran}(\hat{\Phi}(I-P_{\mathcal{S}})) \subseteq \mathcal{S}^{\perp}$. For each density operator ρ whose support is in $\mathcal{S}^{\perp}$, $\rho \leqslant I - P_{\mathcal{S}}$ implies $\Phi(\rho) \leqslant \Phi(I-P_{\mathcal{S}})$. Hence the support of $\Phi(\rho)$ is also in $\mathcal{S}^{\perp}$, i.e., Φ fixes $\mathcal{S}^{\perp}$. Noting that $\dim \mathcal{S}^{\perp} < +\infty$, by step 2, we get the desired result.

($\Leftarrow$): If $\Phi(I-P_{\mathcal{S}}) = I - P_{\mathcal{S}}$, then $\hat{\Phi}(P_{\mathcal{S}}) = P_{\mathcal{S}}$. For each density operator ρ whose support is in $\mathcal{S}$, $\rho \leqslant P_{\mathcal{S}}$ gives $\hat{\Phi}(\rho) = \Phi(\rho) \leqslant \hat{\Phi}(P_{\mathcal{S}}) = P_{\mathcal{S}}$. Therefore the support of $\Phi(\rho)$ is in $\mathcal{S}$, that is to say Φ fixes $\mathcal{S}$.

Step 4. If $\dim \mathcal{S} = +\infty$, $\dim \mathcal{S}^{\perp} = +\infty$, then Φ fixes $\mathcal{S}$ if and only if $\hat{\Phi}(P_{\mathcal{S}}) \leqslant P_{\mathcal{S}}$, which is in turn equivalent to $\hat{\Phi}(I-P_{\mathcal{S}}) \geqslant I - P_{\mathcal{S}}$.

($\Rightarrow$): Using the same argument as in step 3, we know that the support of $\hat{\Phi}(P_{\mathcal{S}})$ is also in $\mathcal{S}$. This implies that $\hat{\Phi}(P_{\mathcal{S}})P_{\mathcal{S}} = P_{\mathcal{S}}\hat{\Phi}(P_{\mathcal{S}}) = \hat{\Phi}(P_{\mathcal{S}})$. Since $\hat{\Phi}$ is unital, i.e., $I = \hat{\Phi}(P_{\mathcal{S}}) + \hat{\Phi}(I-P_{\mathcal{S}})$, we obtain

$$1 = \langle x|\hat{\Phi}(P_{\mathcal{S}})|x\rangle + \langle x|\hat{\Phi}(I-P_{\mathcal{S}})|x\rangle$$

for each unit vector $x \in \mathcal{S}$. Note that $\hat{\Phi}(I-P_{\mathcal{S}}) \geqslant 0$, hence

$$\langle x|\hat{\Phi}(P_{\mathcal{S}})|x\rangle \leqslant 1 = \langle x|P_{\mathcal{S}}|x\rangle.$$

Thus $\hat{\Phi}(P_{\mathcal{S}}) \leqslant P_{\mathcal{S}}$ and so $\hat{\Phi}(I-P_{\mathcal{S}}) \geqslant I - P_{\mathcal{S}}$.

($\Leftarrow$): Let ρ be a density operator whose support is in $\mathcal{S}$. Since $\hat{\Phi}$ is order preserving, $\rho \leqslant P_{\mathcal{S}}$ implies that $\Phi(\rho) = \hat{\Phi}(\rho) \leqslant \hat{\Phi}(P_{\mathcal{S}}) \leqslant P_{\mathcal{S}}$. Hence the support of $\Phi(\rho)$ is in $\mathcal{S}$. □

Remark 2.3. Let Φ be a unital quantum operation on finite dimensional Hilbert space H. The main result in [1] states that Φ fixes a subspace $\mathcal{S}$ if and only if $\Phi(P_{\mathcal{S}}) = P_{\mathcal{S}}$ if and only if $Ran(P)$ is A_k-reducing for all k, where $P_{\mathcal{S}}$ is the projection operator onto $\mathcal{S}$. The following concrete example shows that the inequalities in (iii) of theorem 2.2 really happen, which further illustrates the essential difference between the infinite dimensional case and the finite dimensional case. Let $\{|e_n\rangle\}_{n=1}^{+\infty}$ and $\{|f_n\rangle\}_{n=1}^{+\infty}$ be orthonormal bases of Hilbert space $\mathcal{S}$ and $\mathcal{S}^{\perp}$ respectively. Let $S \in \mathcal{B}(\mathcal{S})$ be the bounded linear operator determined by

$$S|e_n\rangle = |e_{n+1}\rangle, n = 1, 2, \ldots.$$

Then $S^{\dagger}|e_1\rangle = 0, S^{\dagger}|e_n\rangle = |e_{n-1}\rangle (n = 2, \ldots)$. Similarly, define $D \in \mathcal{B}(\mathcal{S}^{\perp})$ by

$$D|f_1\rangle = 0, D|f_n\rangle = |f_{n-1}\rangle (n = 2, 3, \ldots)$$

and then $D^{\dagger}|f_n\rangle = |f_{n+1}\rangle, n = 1, 2, \ldots$. Construct the bounded linear operator U on H by

$$U = \begin{pmatrix} S & |e_1\rangle \otimes |f_1\rangle \\ 0 & D \end{pmatrix},$$

where $|e_1\rangle \otimes |f_1\rangle$ is the rank one operator from $\mathcal{S}^{\perp}$ to $\mathcal{S}$ defined by $|e_1\rangle \otimes |f_1\rangle|x\rangle = \langle x|f_1\rangle e_1$ for every $|x\rangle \in \mathcal{S}^{\perp}$. Then

$$UU^{\dagger} = \begin{pmatrix} SS^{\dagger} + |e_1\rangle\langle e_1| & e_1 \otimes D(f_1) \\ D(f_1) \otimes e_1 & DD^{\dagger} \end{pmatrix}.$$

Note that

$$SS^{\dagger} = P_{\mathcal{S}} - |e_1\rangle\langle e_1|, D(f_1) = 0, DD^{\dagger} = I - P_{\mathcal{S}},$$

so $UU^\dagger = I$. Similarly, $U^\dagger U = I$. Therefore $\Phi(T) = UTU^\dagger$ for all $T \in \mathcal{T}(H)$ is a unital quantum operation. For every density operator ρ whose support is in $\mathcal{S}$,

$$\rho = \begin{pmatrix} \rho_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

according to the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^\perp$. It is easy to check that the support of $\Phi(\rho)$ is also in $\mathcal{S}$. But $\hat{\Phi}(P_\mathcal{S}) = P_\mathcal{S} - |e_1\rangle\langle e_1| < P_\mathcal{S}$, $\hat{\Phi}(I - P_\mathcal{S}) > I - P_\mathcal{S}$.

Remark 2.4. From theorem 2.2, if $\dim \mathcal{S} < +\infty$ or $\dim \mathcal{S}^\perp < +\infty$, then it is easy to see that $\mathcal{S}$ is an invariant subspace of Φ if and only if $\mathcal{S}$ is a reducing space of Φ. But if $\dim \mathcal{S} = \dim \mathcal{S}^\perp = +\infty$, then this equivalence does not hold true. In fact, the example in remark 2.3 shows that $\mathcal{S}$ is an invariant subspace of Φ, but $\mathcal{S}^\perp$ is not an invariant subspace of Φ.

Based on theorem 2.2, we have the following structural theorem for unital quantum operations on infinite dimensional density operators. Here we say an invariant subspace $\mathcal{S}$ of Φ is irreducible if it does not contain any proper invariant subspace of Φ.

Corollary 2.5. *Let $\mathcal{S}$ be a proper closed subspace of H and let Φ be a unital quantum operation on $\mathcal{T}(H)$ which fixes $\mathcal{S}$.*

(a) If $\dim \mathcal{S} < +\infty$, then there exists an orthogonal space decomposition $H = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n \oplus \mathcal{S}^\perp$, where n is a positive integer and each $\mathcal{S}_j$ is an irreducible subspace of Φ. Furthermore, every Kraus operator A_k of Φ can also be decomposed as $\oplus_{j=1}^n A_k^{\mathcal{S}_j} \oplus A_k^{\mathcal{S}^\perp}$, where $A_k^{\mathcal{S}_j} \in \mathcal{B}(\mathcal{S}_j)$, $A_k^{\mathcal{S}^\perp} \in \mathcal{B}(\mathcal{S}^\perp)$. Therefore, Φ has a invariant proper closed subspace if and only if all of its Kraus operators can be simultaneously diagonalized into at least two diagonal blocks.

(b) If $\dim \mathcal{S}^\perp < +\infty$, then there exists an orthogonal space decomposition $H = \mathcal{S} \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n$, where n is a positive integer and each $\mathcal{S}_j$ is an irreducible subspace of Φ. Furthermore, every Kraus operator A_k of Φ can also be decomposed as $\oplus_{j=1}^n A_k^{\mathcal{S}_j} \oplus A_k^{\mathcal{S}}$, where $A_k^{\mathcal{S}_j} \in \mathcal{B}(\mathcal{S}_j)$, $A_k^{\mathcal{S}} \in \mathcal{B}(\mathcal{S})$. It follows that Φ has an invariant proper closed subspace if and only if all of its Kraus operators can be simultaneously diagonalized into at least two diagonal blocks.

(c) If $\dim \mathcal{S} = \dim \mathcal{S}^\perp = +\infty$, then all of its Kraus operators can be simultaneously transformed into upper triangular blocks.

Proof. From theorem 2.2, every Kraus operator A_k of the quantum operation Φ admits a direct sum decomposition $A_k^\mathcal{S} \oplus A_k^{\mathcal{S}^\perp}$, where $A_k^\mathcal{S}$ $(A_k^{\mathcal{S}^\perp})$ is a linear operator on $\mathcal{S}(\mathcal{S}^\perp)$. Since $\sum_k A_k^\mathcal{S} A_k^{\mathcal{S}\dagger} = P_\mathcal{S}$, the quantum operation $\Phi|_{\mathcal{B}(\mathcal{S})}(\cdot) = \sum_k A_k^\mathcal{S}(\cdot)A_k^{\mathcal{S}\dagger}$ is unital. By recursively applying theorem 2.2, we have the desired result in (a).

Part (b) is a direct consequence of remark 2.4 and part (a).

Part (c) is an immediate consequence of theorem 2.2. □

3. Applications of commutativity of operations and preservation of measurement statistics

In this section, we are devoted to characterizing the commutativity of two quantum operations and measurement statistics preserving operations. At the same time, we correct part (a) of theorem 6 in [1].

We firstly correct part (a) of theorem 6 in [1]. Assume $\dim H < +\infty$, let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be quantum operation $T \mapsto \sum_k P_k T P_k$, where P_k are projections satisfying $\sum_k P_k = I$ and $P_k P_{k'} = 0$ whenever $k \neq k'$. It is asserted in [1, Part (a) of theorem 6] that $\phi \circ \Phi(T) = \Phi \circ \phi(T)$ for all $T \in \mathcal{B}(H)$ if and only if every $P_k(H)$ is an invariant subspace of Φ, where Φ is a unital quantum operation on $\mathcal{B}(H)$. However, this conclusion is not right as shown by the following example. Let $|e_1\rangle = \binom{1}{0}$ and $|e_2\rangle = \binom{0}{1}$ be an orthonormal base of $H = \mathbb{C}^2$. Define $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\Phi(T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for every $T \in \mathcal{B}(H)$. Define $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\phi(T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

for every $T \in \mathcal{B}(H)$. It is easy to see that both Φ and ϕ are unital operations. Furthermore,

$$\Phi \circ \phi(T) = \phi \circ \Phi(T) = \langle e_2|T^\dagger|e_2\rangle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \langle e_1|T^\dagger|e_1\rangle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

for every $T \in \mathcal{B}(H)$. But

$$\Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that neither $P_1(H)$ nor $P_2(H)$ is an invariant subspace of Φ, where $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. In fact, what we have is the following.

Theorem 3.1. *Let H be a separable infinite dimensional Hilbert space. Let $\phi : \mathcal{T}(H) \to \mathcal{T}(H)$ be the quantum operation $\rho \mapsto \sum_k P_k \rho P_k$, where P_k are projections satisfying $\sum_k P_k = I$ and $P_k P_{k'} = 0$ whenever $k \neq k'$. Assume that $Ran(A_k) \subseteq P_k(H)$ and $A_k|_{P_k(H)} \neq 0$. Then $\phi \circ \Phi(\rho) = \Phi \circ \phi(\rho)$ for all density operators $\rho \in \mathcal{T}(H)$ if and only if every $P_k(H)$ is an invariant subspace of Φ, where Φ is a unital quantum operation on $\mathcal{T}(H)$.*

In order to give the proof of theorem 3.1. We need the following lemma which was proved in [1] when H is a finite dimensional Hilbert space.

Lemma 3.2. *Let P be a projection on H and $P(H)$ denote the range of the projection operator P on H. If $\dim P(H) < +\infty$ or $\dim P(H)^\perp < +\infty$, then $P\Phi(\rho)P = \Phi(P\rho P)$ for all density operators $\rho \in \mathcal{T}(H)$ if and only if $P(H)$ is an invariant subspace of Φ.*

Proof. $(\Rightarrow)$: Assume $\mathcal{S} = P(H)$, then for any density operator ρ whose support is in $\mathcal{S}$, $P\Phi(\rho)P = \Phi(P\rho P)$ implies $\Phi(\rho) = P\Phi(\rho)P$. It follows that the support of $\Phi(\rho)$ is contained in $\mathcal{S}$. So $\mathcal{S}$ is an invariant subspace of Φ.

$(\Leftarrow)$: If $\mathcal{S} = P(H)$ is an invariant subspace of Φ, then (a) and (b) of corollary 2.5 tell us that each Kraus operator A_k of Φ can be written as $A_k = A_k^{\mathcal{S}} \oplus A_k^{\mathcal{S}^\perp}$ according to the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^\perp$:

$$\begin{aligned} P\Phi(\rho)P &= \sum_k \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_k^{\mathcal{S}} & 0 \\ 0 & A_k^{\mathcal{S}\perp} \end{pmatrix} \rho \begin{pmatrix} A_k^{\mathcal{S}} & 0 \\ 0 & A_k^{\mathcal{S}\perp} \end{pmatrix}^\dagger \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ &= \sum_k \begin{pmatrix} A_k^{\mathcal{S}} & 0 \\ 0 & 0 \end{pmatrix} \rho \begin{pmatrix} A_k^{\mathcal{S}\dagger} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \sum_k \begin{pmatrix} A_k^{\mathcal{S}} P\rho P A_k^{\mathcal{S}\dagger} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \sum_k \begin{pmatrix} A_k^{\mathcal{S}} & 0 \\ 0 & A_k^{\mathcal{S}\perp} \end{pmatrix} \begin{pmatrix} P\rho P & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_k^{\mathcal{S}\dagger} & 0 \\ 0 & A_k^{\mathcal{S}\perp\dagger} \end{pmatrix} \\ &= \Phi(P\rho P) \end{aligned}$$

for all density operators $\rho \in \mathcal{T}(H)$. □

Now, we are in a position to prove theorem 3.1.

Proof of theorem 3.1. $(\Leftarrow)$: We divide the proof into two cases.

Case 1. $\dim P_k(H) < +\infty$ or $\dim P_k(H)^\perp < +\infty$ for every k.
Since every $P_k(H)$ is an invariant subspace of Φ, from lemma 3.2, we have $P_k\Phi(\rho)P_k = \Phi(P_k\rho P_k)$. Therefore $\sum_k P_k\Phi(\rho)P_k = \Phi(\sum_k P_k\rho P_k)$. That is $\phi \circ \Phi(\rho) = \Phi \circ \phi(\rho)$ for all density operators $\rho \in \mathcal{T}(H)$.
Case 2. There exists P_{k_0} such that $\dim P_{k_0}(H) = \dim P_{k_0}(H)^\perp = +\infty$.

Denote $\mathcal{S} = P_{k_0}(H)$, then (c) of corollary 2.5 shows that each Kraus operator A_k of Φ can be written as

$$A_k = \begin{pmatrix} A_{k1} & A_{k2} \\ 0 & A_{k3} \end{pmatrix}$$

according to the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^\perp$. Note that $Ran(A_k) \subseteq P_k(H)$ for all k, so $A_{k_0} = \left(\begin{smallmatrix} A_{k_01} & A_{k_02} \\ 0 & 0 \end{smallmatrix}\right)$, $A_k = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A_{k_3} \end{smallmatrix}\right) (k \neq k_0)$ according to the space decomposition $H = \mathcal{S} \oplus \mathcal{S}^\perp$. Since every $P_k(H)$ is an invariant subspace, from (iii) of theorem 2.2, we have $\hat{\Phi}(P_k) \leqslant P_k$ for all k. Hence

$$\hat{\Phi}(I - P_{k_0}) = \hat{\Phi}\left(\sum_{k \neq k_0} P_k\right) \leqslant I - P_{k_0}.$$

Note that $\hat{\Phi}(I) = I$, so $\hat{\Phi}(P_{k_0}) = P_{k_0}$. From (iii) of lemma 2.1, we know $A_{k_0 2} = 0$. By the same computation as in lemma 3.2, $P_{k_0}\Phi(\rho)P_{k_0} = \Phi(P_{k_0}\rho P_{k_0})$ for all density operators $\rho \in \mathcal{T}(H)$. This implies $P_k\Phi(\rho)P_k = \Phi(P_k\rho P_k)$ for all k. Therefore $\sum_k P_k\Phi(\rho)P_k = \Phi(\sum_k P_k\rho P_k)$.
$(\Rightarrow)$: For $T \in \mathcal{T}(H)$, $\phi \circ \Phi(T) = \Phi \circ \phi(T)$ tells us that

$$\sum_k P_k\left(\sum_s A_s T A_s^\dagger\right)P_k = \sum_s A_s\left(\sum_k P_k T P_k\right)A_s^\dagger.$$

Multiplying P_k from right and left, respectively, we get $A_k T A_k^\dagger = A_k(\sum_s P_s T P_s)A_k^\dagger$ since $Ran(A_k) \subseteq P_k(H)$. Because $A_k|_{P_k(H)} \neq 0$, there exists $|x\rangle \in P_k(H)$ such that $A_k|x\rangle \neq 0$. Choose arbitrarily $|y\rangle \in P_k(H)^\perp$, substituting the rank one operator $T = |x\rangle \otimes |y\rangle$ into this equality, we have $A_k|x\rangle \otimes A_k|y\rangle = 0$. Hence $A_k|_{P_k(H)^\perp} = 0$ and $A_k = \left(\begin{smallmatrix} A_{k1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ according to the space decomposition $H = P_k(H) \oplus P_k(H)^\perp$. Therefore for each density operator ρ whose support is in $P_k(H)$, it is easy to see that $\Phi(P_k\rho P_k) = P_k\Phi(\rho)P_k$. So $P_k(H)$ is an invariant subspace of Φ. $\square$

Remark 3.3. Our assumptions that $Ran(A_k) \subseteq P_k(H)$ and $A_k|_{P_k(H)} \neq 0$ are indispensable to theorem 3.1. Considering the example before theorem 3.1, it is easy to check that $Ran(A_i) \subseteq P_i(H), A_i|_{P_i(H)} = 0, i = 1, 2$. Let

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One can check that $Ran(A_i) \nsubseteq P_i(H), A_i|_{P_i(H)} \neq 0, i = 1, 2$.

In measurement statistics, a POVM measurement is described by a collection $\{E_m\}$ of positive linear operators on H which are called measurement operators. The measurement operators satisfy the completeness equation, $\sum_m E_m^2 = I$. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement, then the probability that result m occurs is given by $p(m) = \mathrm{tr}(E_m^2\rho)$ and the state of the system after the measurement is $\frac{E_m\rho E_m}{\mathrm{tr}(E_m^2\rho)}$ (see [3]). The completeness equation expresses the fact that probabilities sum to one, that is $\sum_m p(m) = 1$. For a given POVM measurement $\{E_m\}$ and a quantum operation Φ, from the point of measurement statistics, Φ is called measurement statistics preserving if $\mathrm{Tr}(E_m\rho) = \mathrm{Tr}(E_m\Phi(\rho))$ hold true for all density operators $\rho \in \mathcal{T}(H)$. Characterizing measurement statistics preserving quantum operations is a fundamental problem. In the following, we will study this question and generalize [1, theorem 6] to the infinite dimensional case and give a unified proof.

Theorem 3.4. *Let $E_1, E_2, \ldots$ be positive operators with $\sum_k E_k^2 = I$, and let $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ be a unital quantum operation. $Tr(E_k\rho) = Tr(E_k\Phi(\rho))$ for all density operators $\rho \in \mathcal{T}(H)$ and for all k if and only if the range of spectral projections $\{E_\lambda^{(k)}\}_{\lambda\in\mathbb{R}}$ of E_k are reducing subspaces of Φ.*

Proof. $(\Rightarrow)$: Note that $\Phi^\dagger(T) = \sum_k A_k^\dagger T A_k$ for all $T \in \mathcal{B}(H)$ which is also a unital quantum operation. Furthermore,

$$\mathrm{tr}(T\Phi(S)) = \sum_k \mathrm{tr}\big(TA_kSA_k^\dagger\big) = \sum_k \mathrm{tr}\big(A_k^\dagger TA_kS\big) = \mathrm{tr}(\Phi^\dagger(T)S)$$

for all $T, S \in \mathcal{T}(H)$. Thus $\mathrm{tr}(E_k\rho) = \mathrm{tr}(E_k\Phi(\rho)) = \mathrm{tr}(\Phi^\dagger(E_k)\rho)$ for all $\rho \in \mathcal{T}(H)$. This implies $\Phi^\dagger(E_k) = E_k$ for every E_k. By direct computing, we get

$$\begin{aligned}\sum_i [E_k, A_i]^\dagger[E_k, A_i] &= \sum_i \big(A_i^\dagger E_k - E_kA_i^\dagger\big)(E_kA_i - A_iE_k) \\ &= \Phi^\dagger\big(E_k^2\big) - \Phi^\dagger(E_k)E_k - E_k\Phi^\dagger(E_k) + E_k^2 \\ &= \Phi^\dagger\big(E_k^2\big) - E_k^2.\end{aligned}$$

So $\Phi^\dagger(E_k^2) \geqslant E_k^2$ for every k. Combining $\sum_k E_k^2 = I$ with $\Phi^\dagger(I) = I$, we have $\Phi^\dagger(E_k^2) = E_k^2$. Hence $\sum_i [E_k, A_i]^\dagger[E_k, A_i] = 0$. This tells us $[E_k, A_i] = 0$. Therefore A_i commutes with the spectral projections $\{E_\lambda^{(k)}\}_{\lambda\in\mathbb{R}}$ of E_k ([21, theorem 2.5.6]). From theorem 2.2, we get the desired result.

$(\Leftarrow)$: From our assumption, it is clear that spectral projections $\{E_\lambda^{(k)}\}_{\lambda\in\mathbb{R}}$ of E_k commute with A_i. So E_k commutes with A_i. Therefore $\mathrm{tr}(E_k\Phi(\rho)) = \mathrm{tr}(E_k\sum_i A_i\rho A_i^\dagger) = \mathrm{tr}(\Phi(E_k\rho)) = \mathrm{tr}(E_k\rho)$. $\square$

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